ON THE EVOLUTION BOUSSINESQ–STEFAN PROBLEM FOR NON-NEWTONIAN FLUIDS

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Abstract. We consider a mathematical model for the coupling of diffusion and convection phenomena in continuous media with change of phase. We describe the case of a dilatant fluid in a three dimensional evolution case, for which the continuity of the temperature yields the existence of a weak solution and we suggest how this approach can be extended to the case of the solidification of a Bingham fluid.

1. Introduction

It is known that in melting or solidification problems the natural convection in the bulk liquid plays an important role, in particular, that the movement of liquid may change considerably the shape of the solid-liquid interface. From the mathematical point of view, it is necessary to couple the Stefan problem with a convective term given by the Boussinesq approximation for the fluid flow.

The main mathematical difficulty when we consider generalized solutions, which seems the natural framework for the general case, lies in the delicate question of the definition of the liquid zone and the formulation of the respective equations of motion. If we normalize the temperature \( \vartheta = \vartheta(x, t) \), \( x \in \Omega \subset \mathbb{R}^N \), \( t \in [0, T] \), in such a way that the liquid region \( \Lambda \) may be defined by

\[
\Lambda = \{ \vartheta > 0 \} = \{(x, t) \in Q : \vartheta(x, t) > 0 \},
\]

we should require this set to be an open subset of \( Q = \Omega \times [0, T] \).

A sufficient requirement for this topological property is the continuity (or at least the lower semicontinuity) of the temperature \( \vartheta \), which is a delicate regularity property for the solution of the Stefan problem with convection. This has been analysed for Newtonian fluids in [3] for \( N = 2 \) and in [6, 7] for \( N = 3 \). For the steady-state case, the problem is much simpler as it was shown in [2] and extended to more general situations in [12, 13] and [15]. In this later work, the stationary Boussinesq–Stefan problem has been considered for general constitutive power-laws both for the temperature and velocity equations, including also laws of non-Newtonian fluids for which the continuity of the temperature was obtained when the product of the respective two powers is greater than the space dimension \( N = 2, 3 \).
In general, it is physically natural to assume the velocity field \( v = v(x,t) \) vanishes in the solid region \( \Sigma = \{ \vartheta < 0 \} \) while in the liquid zone, the Boussinesq approximation for an incompressible fluid is given by

\[
\nabla \cdot v = 0 \quad \text{in} \quad \{ \vartheta > 0 \},
\]

\[
\partial_t v + (v \cdot \nabla) v - \nabla \cdot \mathbf{S} + \nabla p = f(\vartheta) \quad \text{in} \quad \{ \vartheta > 0 \},
\]

where \( \partial_t = \partial/\partial t, \nabla \cdot = \text{div}, \nabla = \text{grad}, \mathbf{S} = (S_{ij}) \) is the viscous stress tensor, \( p \) is the pressure and \( f \) is the density of (buoyancy) forces that may depend on the temperature.

For some non-Newtonian fluids, we may define the constitutive law by

\[
S_{ij} = \nu(\vartheta) \frac{D_{ij}^2}{D_{II}^{\frac{q}{2}}} D_{ij}(v) + \gamma(\vartheta) \frac{D_{ij}(v)}{D_{II}^{\frac{1}{2}}(v)},
\]

if \( D_{II} > 0 \), where \( D_{II}(v) = \frac{1}{2} D_{ij} D_{ij} \) and \( D_{ij} = \frac{1}{2} (\partial v_i / \partial x_j + \partial v_j / \partial x_i) \), with the usual summation convention. Here \( \nu = \nu(\vartheta) \) is a (temperature) dependent viscosity coefficient, \( q > 1 \) a given parameter and \( \gamma = \gamma(\vartheta) \geq 0 \) the plasticity threshold coefficient that, in the Bingham model, corresponds to a rigid motion \( (D_{ij} = 0) \) if and only if \( S_{ij} S_{ij} \leq 2 \gamma^2 \).

When \( \nu \equiv \text{const.} > 0, q = 2 \) and \( \gamma \equiv 0, (1.4) \) corresponds to a Newtonian fluid and (1.3) reduces to the Navier–Stokes equations. If \( 1 < q < 2 \) the fluid is called pseudo-plastic and if \( q > 2 \), the case of interest in this note, it is called a dilatant fluid.

We shall assume an initial condition \( v(x,0) = v_0(x) \) defined on the initial liquid zone \( \{ \vartheta_0 > 0 \} \) and, for simplicity, that the fluid adheres to the boundary, i.e.

\[
v = 0 \quad \text{on} \quad \partial \{ \vartheta > 0 \},
\]

in particular, on the free boundary \( \Phi = \{ \vartheta = 0 \} = \partial \{ \vartheta > 0 \} \cap \partial \{ \vartheta < 0 \} \).

From the energy balance, we can derive the strong formulation of the Stefan problem with convection (see [14], [11])

\[
\partial_t \vartheta + v \cdot \nabla \vartheta = \Delta \vartheta \quad \text{in} \quad \{ \vartheta < 0 \} \cup \{ \vartheta > 0 \},
\]

\[
[\nabla \vartheta]_+^t \cdot \mathbf{n}_x = -\lambda \mathbf{w} \cdot \mathbf{n}_x = \lambda n_t \quad \text{in} \quad \{ \vartheta = 0 \},
\]

where \( (\mathbf{n}_x, n_t) \) is the space-time normal vector to the free boundary \( \Phi = \{ \vartheta = 0 \} \), \( \mathbf{w} \) its velocity and \( \lambda > 0 \) the latent heat. Here the assumption \( v = 0 \) on \( \Phi \) plays an essential role in the equation (1.7). Adding an initial condition \( \vartheta(x,0) = \vartheta_0(x) \) and lateral boundary conditions on \( \partial \Omega \times ]0,T[ \), that may be of Dirichlet, Neumann or mixed type, the formulation of the problem is completed. In fact, for simplicity of presentation, we shall consider only homogeneous Dirichlet data \( \vartheta = 0 \), although the general case can be treated easily in the variational formulation (see, for instance, [14]).

This note is composed of two parts: in the first one we describe the general formulation of the three-dimensional problem for a dilatant fluid, for which an existence result has been given by [16] for the case \( q > N = 3 (\gamma \equiv 0) \); in the second part we suggest a variational inequality formulation for a special case of the solidification of a Bingham flow which can be regarded as an extension to the evolution problem of the earlier approach of [13].
2. The 3-D Boussinesq–Stefan problem for a dilatant fluid

In this section we consider the weak formulation for the problem (1.2)–(1.7) with homogeneous Dirichlet boundary conditions in the case \( q > N = 3 \) and \( \gamma \equiv 0 \). Multiplying (1.6) by a smooth test function, separately in \( \{ \vartheta < 0 \} \) and in \( \{ \vartheta > 0 \} \), integrating by parts and assuming that \( \Phi = \{ \vartheta = 0 \} \) is described by a smooth interface, the jump condition (1.7) implies that we have in the sense of distributions

\[
\partial_t \eta + \mathbf{v} \cdot \nabla \vartheta = \Delta \vartheta \quad \text{in } \mathcal{D}'(Q) ,
\]

where the enthalpy \( \eta \in \beta(\vartheta) \) and \( \beta \) is given by

\[
\beta(s) = \begin{cases} 
\lambda + s, & s > 0 , \\
[0, \lambda], & s = 0 , \\
s, & s < 0 .
\end{cases}
\]

Assuming that \( \vartheta \) is continuous in \( Q_T \), hence \( \{ \vartheta > 0 \} \) is an open subset where we may define solenoidal smooth test functions with support contained in \( \{ \vartheta > 0 \} \), we can also integrate by parts the equation (1.3) with the constitutive relation (1.4). We introduce the initial data \( \eta_0 \in L^\infty(\Omega) \) and \( \vartheta_0 \in C(\Omega) \) and the spaces of appropriate test functions, namely

\[
\mathcal{T}_1 = \left\{ \xi \in H^1(Q) : \xi(T) = 0, \xi = 0 \text{ on } \partial \Omega \times ]0,T[ \right\} , \quad (2.2)
\]

\[
\mathcal{T}(\vartheta) = \left\{ \Psi \in L^q(0,T;V^q) \cap [H^1_0(Q)]^3 : \text{supp } \Psi \subset \{ \vartheta > 0 \} \right\} , \quad (2.3)
\]

\[
\mathcal{T}_{\vartheta_0} = \left\{ \mathbf{r} \in \mathcal{V}(\Omega) : \text{supp } \mathbf{r} \subset \{ \vartheta_0 > 0 \} \right\} , \quad (2.4)
\]

where

\[
\mathcal{V}(\Omega) = \left\{ \mathbf{v} \in [\mathcal{D}(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0 \right\} , \quad H = \text{closure of } \mathcal{V} \text{ in } [L^2(\Omega)]^3 ,
\]

\[
V^q = \text{closure of } \mathcal{V}(\Omega) \text{ in the norm } ||D^{1/2}_H(\mathbf{v})||_{L^q(\Omega)} ,
\]

\[
(\text{in fact, } V^q = \left\{ \mathbf{v} \in [W_0^{1,q}(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \right\} \text{ if } \partial \Omega \text{ is smooth}) .
\]

Then it is not difficult to obtain the following weak or generalized formulation of the Boussinesq–Stefan problem, following the approach suggested in [3]:

\[
\int_Q \left[ -\eta \partial_t \xi - \vartheta \mathbf{v} \cdot \nabla \xi + \nabla \vartheta \cdot \nabla \xi \right] = \int_\Omega \eta_0 \xi(0) , \quad \forall \xi \in \mathcal{T}_1 , \quad (2.5)
\]

\[
\int_{\{ \vartheta > 0 \}} \left[ -\mathbf{v} \cdot \partial_t \Psi - \vartheta (\mathbf{v} \cdot \nabla \Psi) + \nu(\vartheta)[D_H(\mathbf{v})]|^{2/3} D \mathbf{v} : D \mathbf{v} - \mathbf{f}(\vartheta) \cdot \Psi \right] = 0 , \quad \forall \Psi \in \mathcal{T}(\vartheta) , \quad (2.6)
\]

\[
\int_\Omega \mathbf{v}(x,t) \cdot \mathbf{r}(x) \xrightarrow{t \to 0^+} \int_\Omega \mathbf{v}_0(x) \cdot \mathbf{r}(x) , \quad \forall \mathbf{r} \in \mathcal{T}_{\vartheta_0} . \quad (2.7)
\]
These formulations make sense under general assumptions on the data, in particular,

\[ \eta_0 \in \beta(\vartheta_0) \quad \text{for} \quad \vartheta_0 \in C^0(\Omega) \cap L^\infty(\Omega), \quad \mathbf{v}_0 \in [L^2(\Omega)]^3, \quad (2.8) \]

\[ \nu \in C^0(\mathbb{R}), \ \nu \geq \nu_* > 0 \quad \text{and} \quad f \in C^0(\mathbb{R}; \mathbb{R}^3) \quad \text{with} \quad f(0) = 0, \quad (2.9) \]

which allow to obtain the following existence of solution.

**Theorem 1** ([16]). Assuming (2.8)–(2.9), there exists at least one weak solution \((\vartheta, \eta, \mathbf{v})\) of the Boussinesq–Stefan problem (2.5), (2.6), (2.7) for a dilatant fluid with \(q > N = 3\) such that

\[ \vartheta \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(Q) \cap C^0(Q), \quad (2.10) \]

\[ \eta \in L^\infty(\Omega), \ \eta \in \beta(\vartheta) \quad \text{a.e. in} \ Q, \quad (2.11) \]

\[ \mathbf{v} \in L^\infty(0, T; H) \cap L^q(0, T; V^q), \quad \mathbf{v} = 0 \quad \text{a.e. in} \ \{\vartheta < 0\}. \quad (2.12) \]

The proof of this result relies in an approximation technique, which is now standard, and in the crucial step of showing the equicontinuity of the sequence of the approximation temperatures \(\vartheta_\varepsilon\) as the regularization parameter \(\varepsilon \to 0\) in the equation

\[ \partial_t \beta_\varepsilon(\vartheta_\varepsilon) + \mathbf{v}_\varepsilon^* \cdot \nabla \vartheta_\varepsilon = \Delta \vartheta_\varepsilon \quad \text{in} \ Q, \]

where \(\beta_\varepsilon\) is a smooth approximation of the maximal monotone graph \(\beta\). Here \(\mathbf{v}_\varepsilon^*\) is a smooth mollification of \(\mathbf{v}_\varepsilon\), preserving the solenoidal property \(\nabla \cdot \mathbf{v}_\varepsilon^* = 0\), where \(\mathbf{v}_\varepsilon\) is a solution of the penalized equation in the whole domain \(Q\)

\[ \partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon - \nabla \left( \nu(\vartheta_\varepsilon) \left( D_{II}(\mathbf{v}_\varepsilon) \right)^{\frac{2}{q-2}} D(\mathbf{v}_\varepsilon) \right) + \nabla p_\varepsilon + \frac{1}{\varepsilon} \chi_\varepsilon(\vartheta_\varepsilon) \mathbf{v}_\varepsilon = f(\vartheta_\varepsilon) \quad \text{in} \ Q, \]

\(\chi_\varepsilon\) being defined for \(\varepsilon > 0\) by

\[ \chi_\varepsilon(s) = \begin{cases} 1 & \text{if} \ s \leq -2\varepsilon, \\ -s/\varepsilon - 1 & \text{if} \ -2\varepsilon < s < -\varepsilon, \\ 0 & \text{if} \ s \geq -\varepsilon. \end{cases} \]

As shown in [16], the condition \(q > N = 3\) is exactly the critical exponent that allows the required energy and logarithmic estimates required to obtain an implicit modulus of continuity for each \(\vartheta_\varepsilon\), independently of \(\varepsilon\), following the technique of [4, 5] and extended in [18].

**Remark 1.** In the two-dimensional case, Theorem 1 is still valid for all \(q \geq 2 = N\). In fact, as shown in the work [3] for the Stokes problem with \(q = 2\), the limit integrability of the velocity \(|\mathbf{v}| \in L^4(Q)\) can be, in some sense, compensated by the information \(\nabla \cdot \mathbf{v} = 0\), yielding the continuity of the temperature. □

**Remark 2.** In the three-dimensional case \(N = 3\) with \(q = 2\), where the continuity of the temperature cannot be expected (even for linear parabolic equations with insufficient
integrable convective term, see [10]) some topological information about the liquid set has been obtained in [6] (see also [7] for the extension to Navier–Stokes equations). In fact, it is possible to show that \( \partial_z \) is equicontinuous up to a possible singular subset \( \Sigma \), closed in \( Q \) of Hausdorff dimension at most 5/3, whose Hausdorff measure is arbitrarily small. Hence, in this case we only have \( \vartheta \in C(Q \setminus \Sigma) \) and the set \( \Lambda = \{ \vartheta > 0 \} \cap \{ Q \setminus \Sigma \} \) is open and we can still guarantee that the velocity \( \mathbf{v} \) satisfies the motion equations in \( \Lambda \), i.e., (2.6) still holds by replacing \( \{ \vartheta > 0 \} \) by \( \Lambda \) for every test function \( \Psi \) with support in \( \Lambda \). \( \square \)

**Remark 3.** Also in the three-dimensional case \( N = 3 \) with \( q = 2 \) and \( \nu \equiv \text{const.} > 0 \), a further analysis of [19] on the properties of the temperature \( \vartheta \) for the Stefan problem with convection and an additional heat source, with which its continuity cannot be expected, allows to assure the existence of an open domain \( \Lambda \subset Q \), such that

\[ \vartheta > 0 \text{ a.e. in } \Lambda \text{ and } \vartheta \leq 0 \text{ a.e. in } Q \setminus \Lambda. \]

This subset is given by \( \Lambda = \bigcup_{z \in Q^+} O_{\rho_z/2}(z) \), where for each

\[ z \in Q^+ \equiv \left\{ z = (x, t) \in Q : \limsup_{\rho \downarrow 0^+} \frac{1}{\rho^{5/3}} \int_{O_{\rho}(z)} |\mathbf{v}|^{10/3} < \infty, \right\}, \]

\[ \lim_{\rho \downarrow 0^+} \int_{O_{\rho}(z)} \vartheta > 0 \text{ and } \lim_{\rho \downarrow 0^+} \int_{O_{\rho}(z)} [\vartheta - \vartheta_{z, \rho}]^- = 0 \right\}, \]

there exists a number \( \rho_z, 0 < \rho_z \leq \text{dist}_{Q}(z, \partial Q) = \text{the parabolic distance between } z \text{ and } \partial Q, \) with \( \text{ess} \inf_{O_{\rho}(z)} \vartheta > 0 \). Here \( O_{\rho}(z) = \{ y : |y - x| < \rho \} \times [t - \frac{q}{2} t^2, t + \frac{q^2}{2} \} \) denotes the parabolic cylinder and \( \vartheta_{z, \rho} = f_{O_{\rho}(z)} \vartheta \text{ the average of } \vartheta \text{ over } O_{\rho}(z). \) \( \square \)

### 3. Solidification of a Bingham fluid

We consider now the general case of the constitutive law (1.4), which can also be written in the subdifferential form

\[ \mathcal{S} \in \partial J(\vartheta; \mathbf{v}) \quad (3.1) \]

where, for each fixed \( \vartheta \), \( J(\vartheta; \cdot) \) is a convex, lower semicontinuous functional depending on \( \mathbf{v} \) through \( D_{\Pi} \), i.e.

\[ J(\vartheta; \mathbf{v}) = 2 \int_{\{ \vartheta > 0 \}} \frac{1}{q} \nu(\vartheta) D_{\Pi}^{q/2}(\mathbf{v}) + \gamma(\vartheta) D_{\Pi}^{1/2}(\mathbf{v}). \]

(3.2)

An equivalent form of (3.1) is given by the inequality

\[ J(\vartheta; \Psi) - J(\vartheta; \mathbf{v}) \geq \int_{\{ \vartheta > 0 \}} \mathcal{S} : [D(\Psi) - D(\mathbf{v})], \]

and integrating in the liquid zone \( \{ \vartheta > 0 \} \) the equation (1.3) multiplied by \( \Psi - \mathbf{v} \), where \( \Psi \) is any smooth solenoidal vector field with \( \text{supp } \Psi \subset \{ \vartheta > 0 \} \) we easily obtain the following variational inequality

\[ \int_{\{ \vartheta > 0 \}} \left[ \partial_t \mathbf{v} \cdot (\Psi - \mathbf{v}) - \nu \cdot (\mathbf{v} \cdot \nabla \Psi) \right] + J(\vartheta; \Psi) - J(\vartheta; \mathbf{v}) \geq \int_{\{ \vartheta > 0 \}} f(\vartheta) \cdot (\Psi - \mathbf{v}), \]

\[ \forall \Psi, \quad \nabla \cdot \Psi = 0, \text{ supp } \Psi \subset \{ \vartheta > 0 \}. \]

(3.3)
However this formulation requires the time derivative $\partial_t \mathbf{v}$ to have a regularity that in general we cannot prove for this problem. A possible way to overcome this difficult is to introduce a weaker formulation as in [9] (see also [8]), that consists formally to replace $\partial_t \mathbf{v}$ by $\partial_t \Psi$, and can be easily done in the case of a cylindrical domain $Q$. In order to derive that weaker formulation we shall assume

$$\{\vartheta > 0\} = \bigcup_{0 < t < T} \Lambda(t)$$
and each $\Lambda(t) = \{x \in \Omega: \vartheta(x, t) > 0\}$ is smooth.

Then, integrating by parts in time and setting $\mathbf{w} = \Psi - \mathbf{v}$, we have

$$\int_{\{\vartheta > 0\}} \mathbf{w} \cdot \partial_t \mathbf{w} = \int_0^T \int_{\Lambda(t)} \partial_t \frac{|\mathbf{w}|^2}{2}$$

$$= \frac{1}{2} \int_{\Lambda(t)} |\mathbf{w}(T)|^2 - \frac{1}{2} \int_{\Lambda(0)} |\mathbf{w}(0)|^2 - \int_0^T \int_{\Lambda(t)} \frac{|\mathbf{w}|^2}{2} \partial_t G G^{-1}$$

where $G$ denotes the Jacobian of the transformation of $\Lambda(t)$ onto $\Lambda(0) = \{\vartheta_0 > 0\}_0 \subset \Omega$, which is assumed to be smooth for every $t \in [0, T]$.

If we assume, in addition, that $\partial_t G \leq 0$, i.e. the liquid zone decreases in time like in a solidification process, we see that the last integral in (3.4) is nonnegative, and we have

$$\int_{\{\vartheta > 0\}} \partial_t \Psi \cdot (\Psi - \mathbf{v}) + \frac{1}{2} \int_{\{\vartheta > 0\}_0} |\Psi(0) - \mathbf{v}_0|^2 \geq \int_{\{\vartheta > 0\}} \partial_t \mathbf{v} \cdot (\Psi - \mathbf{v}) .$$

Then, from (3.3), $\mathbf{v}$ will satisfy

$$\int_{\{\vartheta > 0\}} \partial_t \Psi \cdot (\Psi - \mathbf{v}) - \mathbf{v} \cdot (\mathbf{v} \cdot \nabla \Psi) + J(\vartheta; \Psi) - J(\vartheta; \mathbf{v}) \geq$$

$$\geq \int_{\{\vartheta > 0\}} \mathbf{f}(\vartheta) \cdot (\Psi - \mathbf{v}) - \frac{1}{2} \int_{\{\vartheta > 0\}_0} |\Psi(0) - \mathbf{v}_0|^2 , \quad \forall \Psi \in \mathcal{I}_0(\vartheta) ,$$

where we define then the space of test functions by

$$\mathcal{I}_0(\vartheta) = \{\Psi \in L^q(0, T; V^q) \cap [H^1(Q)]^3: \Psi = 0 \text{ on } \partial\Omega \times [0, T], \quad \Psi(T) = 0$$

and $\text{supp } \Psi \subset \{\vartheta > 0\} \cup \{\vartheta_0 > 0\}_0 \}$.

We note that the initial condition $\mathbf{v}(0) = \mathbf{v}_0$ is now incorporated in the weak formulation (3.5), which reduces to (2.6) (with the complementary condition (2.7)) in the case $\gamma \equiv 0$, since $J$ given by (3.2) is then differentiable.

The corresponding Boussinesq–Stefan problem in weak form consists of (3.5) and (2.5) for which it is also possible to show the existence of solution, taking in particular into account the criteria for the continuity of the temperature field described in the preceding section.
**Theorem 2.** Assuming (2.8)–(2.9) and $\gamma \geq 0$ to be a continuous function, there exists at least one weak solution $(\vartheta, \eta, v)$ with the properties (2.10), (2.11) and (2.12) of the conduction–convection problem (2.5)–(3.5) for a Bingham fluid with $q \geq 2$ if $N = 2$ and $q > 3$ if $N = 3$.

**Remark 4.** We note that in deriving the weak formulation (3.5) we have assumed that we consider a solidification process, but the existence result of Theorem 2 does not guarantee that the liquid region \{\vartheta(t) > 0\} decreases when time evolves. Therefore a further analysis is required.

**Remark 5.** In fact, taking into account the results of [19], if we replace the definition of \{\vartheta > 0\} in (3.5) by the open set $\Lambda$ defined as in the Remark 3, we can formulate the Boussinesq–Stefan without the requirement of the continuity of the temperature and to obtain the existence of a weak solution also for the case $q \geq 2$ in the three-dimensional case $N = 3$.

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